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RAMANUJAN'S PAPER ON BERNOULLI NUMBERS

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1. Introduction. Ramanujan's first paper [28] deals with properties of the Bernoulli numbers. Judging from the unenthusiastic remarks about it in Appendix I of [29], one would expect to find in it only incomplete proofs of well-known results. In the references for the present paper one finds many examples of theorems being proved and then proved again independently. This is to be expected in such a beautiful area of mathematics. However, Ramanujan's paper does contain new results. Furthermore, most of his proofs are quite satisfactory and his paper is a goldmine of facts about the Bernoulli numbers.

We indicate which of Ramanujan's results are true and which are false. Most of the correct ones are stated in modern notation and early references are given for those results which were known before Ramanujan. We present a proof of the von Staudt-Clausen theorem which was suggested by Ramanujan's incomplete proof of that theorem. We prove a few other results which he stated, but which to our knowledge have never been proved until now.

Formulas designated by parenthesized numbers below correspond to formulas with the same number in Ramanujan's paper [28], usually with some change of notation. Parenthesized letters indicate formulas which do not appear in [28]. Thus (1) below is the first numbered formula in [28], expressed in modern notation, while (A) does not appear in [28].

We use the modern definition

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \qquad (|x| < 2\pi) \qquad \text{(A)}$$

for the Bernoulli numbers. Thus, $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$, $B_5 = 0$, etc. From (A) it is easy to derive

$$x \cot x = \sum_{n=0}^{\infty} (-1)^n B_{2n} \frac{(2x)^{2n}}{(2n)!} \qquad (|x| < \pi), \tag{1}$$

which was Ramanujan's starting point.

The properties of Bernoulli numbers which Ramanujan considered may be divided into two broad categories: (i) recursion formulas and (ii) statements related to the von Staudt-Clausen theorem and Kummer's congruences. We discuss (i) in Section 2 and (ii) in Section 3.

The origins of [28] are preserved in Chapter 5 of Ramanujan's Second Notebook [30]. We make occasional reference to that chapter. Berndt and Wilson [4] have analysed that work.

2. Recursion formula. Recursion formulas for the Bernoulli numbers have been featured in at least a dozen papers [3, 15, 16, 17, 19, 23, 24, 27, 28, 31, 33, 36]. Most such formulas have analogues for the Euler numbers. Using (1), Ramanujan equated coefficients of x^n in three trigonometric identities to obtain

$$\sum_{j=0}^{n} \binom{2n+1}{2j} 2^{2j} B_{2j} = 2n+1 \qquad (n \geqslant 0), \tag{2}$$

$$\sum_{j=0}^{n-1} \binom{2n}{2j} B_{2j} = n \qquad (n \geqslant 1), \tag{3}$$

and

$$\sum_{j=0}^{n} \binom{2n+1}{2j} B_{2j} = \frac{2n+1}{2} \qquad (n \geqslant 1). \tag{4}$$

A formula equivalent to (2) appears on page 132 of Euler [13] and again on pages 264–265 of [14]. Euler's formula gives a recursion for the rational numbers $\zeta(2n)/\pi^{2n} = (-1)^{n-1} 2^{2n-1} B_{2n}/(2n)!$, but its equivalence to (2) is clear. Formula (3) was proved by Jacobi on page 265 of [17]. It is equivalent to the formula $\sum_{j=0}^{n-1} \binom{n}{j} B_j = 0$, which is easily deduced from (A). On page 98 of [5], Jacob Bernoulli used (4) to compute

$B_8 = -1/30$. He obtained (4) by setting $m = 1$ and $c = 2n$ in the well-known formula

$$\sum_{r=1}^{m} r^c = \frac{1}{c+1} m^{c+1} + \tfrac{1}{2} m^c + \frac{1}{c+1} \sum_{k=2}^{c} \binom{c+1}{k} B_k m^{c+1-k}, \tag{B}$$

valid for positive integers c and m, which he had just presented. On page 3 of [36], Stern obtained (2), (3), and (4) from identities involving the function $x/(e^x - 1)$ and its derivatives. Of course, Ramanujan's proofs of these identities (and of all the recursion formulas) are correct.

Formulas (2) and (4) combine to give the recursion

$$\sum_{j=1}^{n} \binom{2n+1}{2j} (2^{2j} - 2) B_{2j} = 0 \qquad (n \geqslant 1) \tag{C}$$

for the coefficients in the expansion

$$x \csc x = \sum_{j=0}^{\infty} (-1)^{n-1} (2^{2n} - 2) B_{2n} \frac{x^{2n}}{(2n!)}. \tag{D}$$

From the identity $(x \cot x)^2 = -x^2(1 + d(\cot x)/dx)$, Ramanujan obtained

$$\sum_{j=1}^{n-1} \binom{2n}{2j} B_{2j} B_{2n-2j} = -(2n+1) B_{2n} \qquad (n \geqslant 2), \tag{5}$$

which appears on pages 421–422 Euler [12] and again on pages 265–266 of [14]. The formula is Entry 18 in Chapter 5 of Ramanujan's Second Notebook [30]. The "similar result," obtained from $d(\tan x)/dx = 1 + \tan^2 x$, is

$$\sum_{j=1}^{n-1} \binom{2n}{2j} (2^{2j} - 1)(2^{2n-2j} - 1) B_{2j} B_{2n-2j} = (1 - 2n)(2^{2n} - 1) B_{2n} \quad (n \geqslant 2),$$

which is given on pages 272–273 of Euler [14]. A similar formula,

$$\sum_{j=1}^{n-1} \binom{2n}{2j} (2^{2j} - 1) B_{2j} B_{2n-2j} = -(2^{2n} - 1) B_{2n} \qquad (n \geqslant 2),$$

is proved on page 460 of Meyer [24].

In the recursion formulas above, each Bernoulli number is made to depend on all the preceding (non-zero) ones. One of Ramanujan's goals was to give formulas for the Bernoulli numbers which require less computation. In Sections 3, 4, 18 and 17 of [28] he gave recursion formulas with gaps of 4, 6, 8 and 10, respectively, in the subscripts. The first of these is

$$\sum_{j=0}^{[n/2]}(-1)^j\binom{2n+2}{4j+2}2^{n-2j}B_{2n-4j}=(-1)^{[n/2]}(n+1)/2, \tag{8}$$

where $[x]$ is the greatest integer $\leqslant x$. This formula and those which follow in this section are valid for $n \geqslant 1$. Formula (8) was proved by Knar on pages 455–456 of [19]. The "analogous result" from $\tan \frac{1}{2}x \pm \tanh \frac{1}{2}x$ is

$$\sum_{j=0}^{[n/2]}(-1)^j\binom{2n}{4j}2^{n-2j+1}(1-2^{2n-4j})B_{2n-4j}=(-1)^{[(n+1)/2]}2n. \tag{E}$$

Ramanujan gave four recursion formulas with a gap of 6:

$$\sum_{j=0}^{[n/3]}\binom{2n+3}{6j+3}B_{2n-6j}=\frac{2n+3}{3H_n}, \tag{12}$$

where $H_n=-2$ if $n\equiv 2 \pmod 3$, and $H_n=1$ otherwise,

$$4(1-2^{2n})B_{2n}+3\sum_{j=1}^{[n/3]}\binom{2n}{6j}(1-2^{2n-6j})B_{2n-6j}=-\frac{2n}{H_n}, \tag{36}$$

$$\sum_{j=0}^{[n/3]}\binom{2n+3}{6j+3}2^{6j}(1-2^{2n-6j-1})B_{2n-6j}=\frac{2n+3}{24}(3(-3)^n+1), \tag{37}$$

and

$$\sum_{j=0}^{n}\binom{6n+2}{6j}B_{6j}B_{6n+2-6j}=-\frac{6n+1}{3}. \tag{13}$$

The coefficients in the recursion formulas with gaps larger than 6 are best defined by linear recursion formulas. Ramanujan gave explicit formulas for the coefficients of the formulas with gaps 8 and 10, but then he had to consider 4 or 5 cases. It is more concise to define the coefficients as follows. Define sequences of integers $\{d_n\}$ and $\{e_n\}$ by

n	0	1	2	3	4	5	6	7	8	9
d_n	2	0	3	10	14	−12	−99	−338	−478	408
e_n	2	−1	11	29	−29	123	−199	1364	3571	−3571

$$d_n=-34d_{n-4}-d_{n-8} \qquad (n\geqslant 8)$$

and

$$e_n=123e_{n-5}-e_{n-10} \qquad (n\geqslant 10).$$

For $n\geqslant 0$, define $f_n=1+e_n$ if $(5,n)=1$, and $f_n=(e_n-3)/2$ if $(5,n)=5$. (The f_n are not all integers.)

Then Ramanujan's formulas (43)–(46) may be written simply

$$\sum_{j=0}^{[n/4]}\binom{2n+4}{8j+4}2^{n+1-2[\frac{n+1}{4}]-2j}d_{4j+2}B_{2n-8j}=(-1)^{[n/2]}(n+2)d_{n+2} \tag{F}$$

for $n\geqslant 0$. Formulas (38)–(42) together are equivalent to

$$\sum_{j=0}^{[n/5]}\binom{2n+5}{10j+5}f_{5j+2}B_{2n-10j}=\frac{2n+5}{5}f_{n+2} \tag{G}$$

for $n\geqslant 0$.

The trigonometric identities from which Ramanujan derives these results may be proved by induction either from the identities

$$-2\sin A\sin B=\cos(A+B)-\cos(A-B)$$

$$2\sin A\cos B=\sin(A+B)+\sin(A-B)$$

$$2\cos A\cos B=\cos(A+B)+\cos(A-B)$$

or by changing sine and cosine into complex exponentials. These identities for the product of $\sin \omega x$, where ω runs over the nth roots of unity, have their origins in Kronecker [21].

Most of the recursion formulas with gaps appear in van den Berg [3] or in Haussner [16]. See Lehmer [23] for concise proofs of all but (13). All three of these authors proved general recursions with a gap of $2g$, namely,

$$\sum_{j=0}^{[n/g]}\binom{2n+g}{2gj+g}2^{2n}c(g,j)B_{2n-2gj}=(2n+g)c'(n,g), \tag{H}$$

where $c(g,j)$ and $c'(n,g)$ are certain sums involving the $2g$th roots of unity. Special cases of (H) are (2), (8), (12), (F) and (G) with $g=1, 2, 3, 4, 5$, respectively. Formulas (E) and (36) come from the analogue of

(H) with the Bernoulli numbers replaced by the coefficients in the MacLaurin series for $x \tan x$. Another analogue of (H), for the coefficients in (D), yields (37). All three authors proved both analogues as well as another one for the Euler numbers. Formulas (8), (E), (12), (36), (37), (F) and (G) appear on pages 119, 159, 125, 161, 163, 128 and 138, respectively, of van den Berg [3]. On page 809 of [16], Haussner gave five distinct formulas with a gap of 4, including (8) and (E). Formulas (12), (36) and (37) are on pages 798, 794 and 801, respectively, of [16]. Haussner stated (H) on page 798 of [16] and gave the coefficients c and c' explicitly for $g = 1$, 2 and 3, and in terms of the hypergeometric function for $g = 4$, 5 and 6. Riesel [31] proved (8) and (12).

I could not find (13) in any reference. Apparently, it is truly due to Ramanujan. Unlike the other formulas with gaps larger than 2, it does not seem to extend to an infinite sequence of formulas, one for each gap $2g$ (which would begin with (5), perhaps).

In Section 5, Ramanujan computed $B_2, B_4, \ldots, B_{40}$ as fractions via (12). All values are correct. The same numbers are given in his Second Notebook [30]. While (12) provides a simple, easy method for calculating the first few Bernoulli numbers, it becomes inefficient when used to compute hundreds of them because of the great size of the binomial coefficients. A faster method is described by Knuth and Buckholtz [20]. Some early computations of Bernoulli numbers were done by Jacob Bernoulli (page 97 of [5]), Euler (page 421 of [12] and pages 266–267 of [14]), Ohm [26], and J.C. Adams [1].

3. Number theoretic properties of Bernoulli numbers. In Sections 6 through 15 of [28], Ramanujan considered properties of Bernoulli numbers related to the von Staudt-Clausen theorem and J.C. Adams' theorem. Before examining his work we state these theorems and some related ones which we will need. The first one is proved on page 384 of Borevich and Shafarevich [6]. See pages 257–258 and 261 of Uspensky and Heaslet [38] for proofs of Theorems 1 and 2. Also see [7], [8], [18] and [32].

THEOREM 1. (*von Staudt* [35] *and Clausen* [10]). *For* $n \geqslant 1$,

$$G_{2n} = B_{2n} + \sum_{p-1|2n} 1/p$$

is an integer. In particular, the denominator of B_{2n} *(in lowest terms) is exactly the product of those primes* p *for which* $p-1 \mid 2n$.

THEOREM 2. (*J.C. Adams* [2]). *If* $p-1 \mid 2n$ *and* $p^r \mid 2n$ *for some* $r \geqslant 1$, *then* p^r *also divides the numerator of* B_{2n}.

Now write $B_{2n} = P_{2n}/Q_{2n}$, with relatively prime integers $Q_{2n} > 0$ and P_{2n}. Also, set $S_n(N) = \sum_{a=0}^{n-1} a^n$. The next theorem follows easily from (*B*) and Theorem 1. See pages 258–260 of Uspensky and Heaslet [38] for a proof. Theorem 4 is proved on page 385 of [6] and on pages 264–266 of [38].

THEOREM 3. *For all positive integers* n *and* N, $S_{2n}(N)\, Q_{2n} \equiv NP_{2n}$ *(mod* N^2*).*

THEOREM 4. (*Kummer* [22]; *see Nielsen* [25]). *Let* $p \geqslant 5$ *be prime. Write* $b_m = B_m/m$. *Then* $b_{2n+p-1} \equiv b_{2n} \pmod p$ *provided* $n \geqslant 1$ *and* $p-1 \mid 2n$.

THEOREM 5. (*Kummer* [22]; *see Nielsen* [25], *Stern* [37], *Knuth and Buckholtz* [20]). *Let* $p \geqslant 3$ *be prime. Write* $t_m = 2^m(2^m-1)\, B_m/m$. *Then* $t_{2n+p-1} \equiv t_{2n} \pmod p$ *for all* $n \geqslant 1$.

In Ramanujan's statement (14), it is true that B_n is a fraction, but it is false that the numerator of B_n/n in lowest terms is prime. Riesel remarked [31] that the numerator of $B_{20}/20$ is $174611 = 283.617$. The earliest factorization of Bernoulli numerators (including that of B_{20}) was done by Ohm [26]. See Table 2 of [39] for the complete factorizations up to B_{60}.

Result (15) is a simple consequence of the von Staudt-Clausen theorem.

Results (16) and (17) are true and were known to Euler (see page 495 of [13]; also see page 258 of Nielsen [25]).

The first assertion of (18) is J.C. Adams' theorem. The second assertion is false, as Riesel [31] has noted, with the same counterexamples as (14).

In example (a) following (18), the numerators of B_{12} and B_{36} happen to be prime, but that of B_{24} is 103.2294797. (See [39].)

Example (b) is true except that the numerator of $B_{22}/11$ is 131.593. (See Ohm [26].)

Again in example (c), the numerator of B_{20} is composite. The second assertion is the case $r=1$ of J.C. Adams' theorem.

Section 7 is completely correct except that 15 does divide the denominator of B_{28}. Perhaps Ramanujan meant to say that the denominators of $B_{14}, B_{28}, \ldots$ are not *all* divisible by 15. While this assertion is valid, there do exist composite numbers c, called *Carmichael numbers*, which divide the denominator of B_{2ck} for every $k \geqslant 1$. (Carmichael numbers are squarefree composite numbers c for which $p-1 \mid c-1$ for every prime p dividing c. See Carmichael [9]. The smallest Carmichael number is 561.)

Result (19) is part of Theorem 1 and (20) is very similar to another part. However, (20) is false for some even n because $-1+\sum_{p-1 \mid n} p^{-1}$ can exceed 1: Since the sum of the reciprocals of all primes diverges, there exist primes $p_1, \ldots, p_k$ so that $\sum_{i=1}^{k} p_i^{-1} > 2$. Then (20) fails for $n =$ the least common multiple of $p_1 - 1, \ldots, p_k - 1$. I do not know the least n for which (20) fails.

Result (21) is a simple consequence of Theorem 1. We formulate (22) and (23) in the following three theorems.

THEOREM 6. *If 4 divides n, then 20 divides* $B_n + 1/30$.

Proof. Setting $N=4$ in Theorem 3 gives $(1^n + 2^n + 3^n)\, Q_n \equiv 4P_n \pmod{16}$, or $2Q_n \equiv 4P_n \pmod{16}$. Thus, $30P_n \equiv 15Q_n \equiv -Q_n \pmod{8}$, whence $B_n + 1/30 \equiv 0 \pmod{4}$. Repeat this procedure with $N=5$ in

Theorem 3. Since $1^n + 2^n + 3^n + 4^n \equiv -1 \pmod{5}$, we have $5P_n + Q_n \equiv 0 \pmod{25}$ or $30P_n + 6Q_n \equiv 30P_n + Q_n \equiv 0 \pmod{25}$, so $B_n + 1/30 \equiv 0 \pmod{5}$.

THEOREM 7. *If* $(n, 4) = 2$, *then 5 divides* $B_n/n - 1/12$.

Proof. By Theorem 4, $B_n/n \equiv B_2/2 = 1/12 \pmod{5}$.

THEOREM 8. *If n is any non-negative integer, then*

$$\frac{2(2^{4n+2}-1)B_{4n+2}}{2n+1},\ \frac{-2(2^{8n+4}-1)B_{8n+4}}{2n+1},\ \frac{-2(2^{8n+4}-1)B_{16n+8}}{2n+1}$$

are all integers of the form $30k+1$.

Proof. It is clear from (15) and (16) that the first two numbers are odd integers and that $2^{8n+4}+1$ times the third number is an odd integer. It suffices to show that each number is $\equiv 1 \pmod{15}$ and that the third number is an integer.

Apply Theorem 5 with $p=3$ and then with $p=5$ to get

$$\frac{2^{4n+2}(2^{4n+2}-1)B_{4n+2}}{4n+2} = t_{4n+2} \equiv t_2 = 1 \pmod{3.5}.$$

Since $2^{4n} \equiv 1 \pmod{15}$, we find $2(2^{4n+2}-1)B_{4n+2}/(2n+1) \equiv 1 \pmod{15}$. The proof for the second number is similar: Theorem 5 yields

$$\frac{2^{8n+4}(2^{8n+4}-1)B_{8n+4}}{8n+4} = t_{8n+4} \equiv t_4 = -2 \pmod{15}.$$

Using $2^{8n} \equiv 1 \pmod{15}$, we get

$$\frac{2(2^{8n+4}-1)B_{8n+4}}{2n+1} \equiv -1 \pmod{15}.$$

Likewise,

$$\frac{-(2^{16n+8}-1)B_{16n+8}}{2n+1} \equiv 1 \pmod{15}.$$

But $2^{8n+4}+1 \equiv 2 \pmod{15}$. If the third number were not an integer, then some prime p dividing $2^{8n+4}+1$ would have to divide the denominator of $B_{16n+8}/(2n+1)$. By Theorem 2, the primes in the denominator of $B_{1m}/2m$ are the same as those in the denominator of B_{2m}. Hence, $p-1 \mid 16n+8$ by Theorem 1. But if $p \mid 2^{8n+4}+1$, then $16 \mid p-1$, which is a contradiction. Hence, the third number is an integer.

The conclusions of Section 10 are correct, but the reasoning in the last sentence is faulty; recall example (b) above. (B_{23} should be B_{22}.)

Section 11 is correct through (27), providing n is even and $\geqslant 2$. As the editor notes in Appendix I of [29], the formulas after (27) are invalid since the series in the integrand diverges when $x \geqslant n$.

In Section 12 Ramanujan offers some "proof" for (20), (21) and (16). However, he works with divergent series incorrectly and does not prove that certain ones have integer coefficients. Nevertheless, his proofs can be repaired, though with much effort. To illustrate the repairs needed, we will give a complete proof of the von Staudt-Clausen theorem from which (28) can be deduced easily. Schwering [34] has given a proof much like ours.

Proof of Theorem 1. For $x \geqslant 0$ and positive integers n define $x^{(-n)} = ((x+1)(x+2)\ldots(x+n))^{-1}$ and $x^{(0)} = 1$. Write $d_j = j!x^{(-j)}/x$ for $j \geqslant 0$ and a fixed $x > 0$. Then $d_j - d_{j-1} = (j-1)!x^{(-j)}$, $\lim_{j\to\infty} d_j = 0$, and

$$\sum_{j=1}^{\infty} (j-1)!x^{(-j)} = \sum_{=1}^{\infty} (d_j - d_{j-1}) = d_0 = \frac{1}{x}.$$

Replace x by $x+1$, divide by $x+1$, and use $(x+1)^{(-j)}/(x+1) = x^{(-j-1)}$ to get

$$(x+1)^{-2} = \sum_{j=1}^{\infty} (j-1)!x^{(-j-1)} = \sum_{j=1}^{\infty} \frac{(j-1)!}{-j} ((x+1)^{(-j)} - x^{(-j)}).$$

Now let x be a fixed positive integer. We have

$$\sum_{n=1}^{x} n^{-2} = \sum_{n=1}^{x} \sum_{j=1}^{\infty} \frac{(j-1)!}{-j} (n^{(-j)} - (n-1)^{(-j)})$$

$$= \sum_{j=1}^{\infty} \frac{(j-1)!}{-j} (x^{(-j)} - 0^{(-j)}),$$

$$= -\sum_{j=1}^{\infty} \frac{(j-1)!}{j} x^{(-j)} + \sum_{j=1}^{\infty} j^{-2}.$$

Hence,

$$\sum_{n=1}^{\infty} (x+n)^{-2} = \sum_{j=1}^{\infty} \frac{(j-1)!}{j} x^{(-j)}. \tag{J}$$

We now apply the Euler-MacLaurin sum formula to the left side of (J). The result is a special case of Entry 26 of Chapter 5 of Ramanujan's Second Notebook [30], and was justified fully by Berndt and Wilson [4]. The formula is

$$\sum_{n=1}^{\infty} (x+n)^{-2} = \sum_{m=0}^{M} B_m x^{-m-1} + 0(x^{-M-2}) \tag{K}$$

as $x \to \infty$, for each positive integer M. Comparison of (J) and (K) yields

$$\sum_{j=1}^{\infty} \frac{(j-1)!}{j} x^{(-j)} = \sum_{m=0}^{M} B_m x^{-m-1} + 0(x^{-M-2}) \tag{L}$$

as $x \to \infty$. Now we expand $x^{(-j)}$ in a power series in $1/x$. The factor $(x+r)^{-1}$ becomes

$$\frac{1}{x+r} = \frac{1}{x}\left(1 - \frac{r}{x} + \frac{r^2}{x^2} - \ldots + \left(-\frac{r}{x}\right)^n\right) + 0(x^{-M-2}). \tag{M}$$

To prove Theorem 1 for a particular B_m, fix $M \geqslant m$. Since $x^{(-j)} = 0(x^{-j})$, we may truncate the sum on the left side of (L) at $j = M+1$ without altering the truth of (L). Make the substitution (M), multiply out, and discard powers of $1/x$ higher than the $M+1$ power. It follows from (L) that the coefficient of x^{-m-1} in the polynomial in $1/x$ on the left must be B_m, for $m = 0, 1, \ldots, M$. The polynomial in $1/x$ on the right side of (M) has integral coefficients: hence; so does that which replaces $x^{(-j)}$. When j is composite and $j \neq 4$, j divides $(j-1)!$, and so the term

$$\frac{(j-1)!}{j} x^{(-j)} \tag{N}$$

of (L) contributes no fraction to B_m. The same is true for the term (N) with $j = 1$. When j is prime, $(j-1)! \equiv -1 \pmod{j}$ by Wilson's theorem, and $\prod_{i=1}^{j}(x+i)\equiv(x^j-x) \pmod{j}$. Thus, the fractional contribution of (N) to B_m is exactly the same as that of

$$\frac{-1}{j(x^j-x)} = \frac{-1}{jx}(x^{-j+1}+x^{-2(j-1)}+x^{-3(j-1)}+\dots), \tag{P}$$

namely, $-1/j$ when $j-1\mid m$ and 0 otherwise. Note that for $j=2$, the term (N) contributes $\frac12$ to B_m for every $m\geqslant 1$, even for odd m. Now $B_1=-\frac12$, but $B_{2k+1}=0$ for $k\geqslant 1$, so we had better find another contribution of $\frac12$ to these Bernoulli numbers. This is exactly what is provided by the remaining term, that with $j=4$. For $(4-1)!/4=3/2$ and $(x+1)(x+2)(x+3)(x+4)\equiv(x^3-x)(x+2) \pmod{4}$, so that the fractional contribution of (N) with $j=4$ is the same as for

$$\frac{1}{2(x^3-x)(x+2)} = \frac{1}{2x^4}\left(1+\frac{1}{x^2}+\frac{1}{x^4}+\dots\right)\left(1-\frac{2}{x}+\frac{4}{x^2}-\dots\right),$$

namely, $\frac12$ when m is odd and $\geqslant 3$, and 0 otherwise. Hence,

$$G_m = B_m + \sum_{p-1\mid m} 1/p$$

is an integer for $m=1$ and for even $m\geqslant 0$, and B_m is an integer (0, of course) for odd $m\geqslant 3$. This proves Theorem 1.

To prove (28), very little additional work is needed. In (K), move the terms with $B_0=1$ and $B_1=-\frac12$ to the left side:

$$\frac{1}{2x^2}+\sum_{n=1}^{\infty}(x+n)^{-2}-\frac{1}{x}=\sum_{k=1}^{K}B_{2k}x^{-2k-1}+0(x^{-2K-2})$$

$$=\sum_{k=1}^{K}(G_{2k}-1)x^{-2k-1}+\sum_{k=1}^{K}\left(1+\sum_{p-1\mid 2k}\frac{-1}{p}\right)x^{-2k-1}+0(x^{-2K-2}). \tag{Q}$$

Now

$$\sum_{k=1}^{K}x^{-2k-1}=\frac{1}{(x^3-x)}+0(x^{-2K-2})$$

while, for odd primes p,

$$\frac{1}{p}\sum_{\substack{k=1\\ p-1\mid 2k}}^{K}x^{-2k-1}=\frac{1}{p(x^p-p)}+0(x^{-2K-2})$$

by (P). Move the second sum on the right side of (Q) to the left side and make the changes just indicated. Then we have

$$\frac{1}{2x^2}+\sum_{n=1}^{\infty}(x+n)^{-2}-\frac{1}{x}-\frac{1}{6(x^3-x)}+\sum_{\substack{p>4\\ p\text{ prime}}}\frac{1}{p(x^p-p)} = \sum_{k=1}^{K}(G_{2k}-1)x^{-2k-1}+0(x^{-2K-2}) \tag{28}$$

as $x\to\infty$.

The $-1/6$ arises as $-1+\frac12$ (from $p=2$) $+1/3$ (from $p=3$). The first seven numbers $G_{2k}-1$ vanish. The next few are 1, -7, 55, -529, 6192, -86580, 1425517. See Table 3 of Knuth and Buckholtz [20] for more values up to G_{250}.

In Section 13, the formula

$$\log_{10}|B_n|\approx(n+\tfrac12)\log_{10}n-1.2324743503n+0.7001199298+0.0361912068/n, \tag{33}$$

valid for large even n, comes from the approximation

$$\ln|B_n|\approx(n+\tfrac12)\ln n-n(1+\ln 2\pi)+\tfrac12\ln 8\pi+\frac{1}{12n},$$

which follows from

$$\zeta(n)=\sum_{k=1}^{\infty}k^{-n}=\prod_{p\text{ prime}}(1-p^{-n})=\frac{(-1)^{n/2-1}(2\pi)^nB_n}{2n!} \tag{24}$$

(for even $n\geqslant 2$), the estimate $\zeta(n)\approx 1$, which holds for large n, and Stirling's formula.

The results of Section 14 are correct.

Following Ramanujan in Section 15, let us write $B_{444}=N/D$, where N and D are relatively prime integers and $D>0$. Then

$$D=2\cdot3\cdot5\cdot7\cdot13\cdot149\cdot223=90709710$$

by the von Staudt-Clausen theorem, $N < 0$, and $-N =$

1588451356825836373100504143782145957991745395344130355584579271802606553073439389459507761417572615243777557855783897943907134004242049552240642874217114210308705949074744929596932632241358688262813726174958748320025797188191224303535908875129465586234092629175800592249211106320935394054787272761311671502199224520443098879151573888801771623192883526328927129807622792528335792700318705689540933290598347707055522491251314640804320693900281215224262309217387438431268666875059083616022905739177343532424279695920452691897028237708410245697642781733124304062104070106669202262888345577753573178251769219880004565578924500260455941319286 57

which has 639 digits. The integral part Ramanujan considers is $[-B_{444}] =$

1751137068816377401163011262831890828436939546322141648986177192940652718516506545395755053585302626635866830415160513625175445940949485509589483721441885442128587436502056170138044011400398613739518362201342594084203302912017074065564853808147612711297309066204716035904106697297735755684670061907503303694821916203271941660397479925817992618352363008645128838000064078347685079142229709361224209520660442052515904760927078708600443038570856584419267895547096759886771403800751690376067851381923098777675286316321071822377430571748165060803774829983739382058481612460994553536642831045910886257400329246780797398414397976362790644

with 631 digits. Ramanujan's statement about $B_{444} + 1/30$ is (22), which we proved in Theorem 6. Everything Ramanujan says in Section 15 is true except his remark that $-N/37$ is prime. In fact 4457 divides this quotient. Moreover, the cofactor $-N/(37 \cdot 4457)$ is a composite number of 633 digits.

REFERENCES

1. ADAMS, JOHN COUCH, On the calculation of Bernoulli's numbers up to B_{62} by means of Staudt's Theorem, *Report of the British Association* (1877) 8-14.

2. ADAMS, JOHN COUCH, Table of the first sixty-two numbers of Bernoulli, *J. Reine Angew. Math.* 85 (1878) 269–272.

3. VAN DEN BERG, F.J., Over periodieketerugloopende betrekkingen tusschen de coëfficiënten in de ontwikkeling van functiën; meer in het bijzonder tusschen de Bernoulliaansche en ook tusschen eenige daarmede verwante coëfficiënten, *Verslagen en Mededeelingen der Koninklijke Akademie van Wetenschappen* (2) 16 (1881) 74–176.

4. BERNDT, BRUCE C. and B.M. WILSON, Chapter 5 of Ramanujan's second notebook, Proceedings of a number theory conference in honor of Emil Grosswald, (to appear).

5. BERNOULLI, JACOB, *Ars Conjectandi*, Basel (1713). Reprinted on pages 107–286 in volume 3 of *Die Werke von Jakob Bernoulli*, Birkhäuser Verlag, Basel (1975).

6. BOREVICH, Z.I. and I.R. SHAFAREVICH, *Number Theory*, Nauka Pub., Moscow (1964); Academic Press, New York (1966).

7. CARLITZ, LEONARD, Some congruences for the Bernoulli numbers, *Amer. J. Math.* 75 (1953) 163–172.

8. CARLITZ, LEONARD, The Staudt-Clausen theorem, *Math. Mag.* 34 (1960/61) 131–146.

9. CARMICHAEL, ROBERT D., On composite numbers P which satisfy the Fermat Congruence $a^{P-1} \equiv 1 \bmod P$, *Amer. Math. Monthly* 19 (1912) 22–27.

10. CLAUSEN, THOMAS, Theorem, *Astronomische Nachrichten* 17 (1840) 351–352.

11. ELY, G.S., Bibliography of Bernoulli's numbers, *Amer. J. Math.* 5 (1882) 228–235.

12. EULER, LEONHARD, *Institutiones calculi differentialis*, Academiae Imperialis Scientiarum Petropolitanae, (1755). Reprinted in *Opera Omnia* I.10, Teubner, Leipzig and Berlin (1923).

13. EULER, LEONHARD, De summis serierum numeros Bernoullianos involventium, *Novi commentarii academiae scientiarum Petropolitanae* 14 (1769): I, 1770, 129–167. Reprinted in *Opera Omnia* I.15, Teubner, Leipzig and Berlin (1927).

14. EULER, LEONHARD, De seriebus potestatum reciprocis methodo nova et facillima summandis, *Opuscula analytica* 2 (1785) 257–274. Reprinted in *Opera Omnia* I.15, Teubner, Leipzig and Berlin (1927).

15. GÖPEL, A., Einige Bemerkungen zu der Abhandlung Nr. IV in diesem Hefte (see [33] below) über Recursionsformeln für die Bernoullischen Zahlen, *Archiv der Mathematik und Physik* (1) 3(1843) 64–67.

16. HAUSSNER, ROBERT, Zur Theorie der Bernoulli' schen und Euler' schen Zahlen, *Nachrichten von der Königlichen Gesellschaft der Wissenschaften and der Georg-Augusts-Univerität zu Göttingen* (1893) 777–809.

17. JACOBI, C.G.J., De usu legitimo formulae summatoriae Maclaurinianae, *J. Reine Angew. Math.* 12 (1834) 263–272.

18. JOHNSON, WELLS, *p*-adic proofs of congruences for the Bernoulli numbers, *J. Number Theory* 7 (1975) 251–265.

19. KNAR, Entwickelung den vorzüglichsten Eigenschaften einiger mit den goniometrischen zunächst verwandten Functionen, *Archiv der Mathematik und Physik* (1) 27 (1856) 365–470.

20. KNUTH, DONALD E, and THOMAS J. BUCKHOLTZ, Computation of tangent, Euler, and Bernoulli numbers, *Math. Comp.* 21 (1967) 663–688.

21. KRONECKER, LEOPOLD, Sur quelques fonctions symétriques et sur les nombres de Bernoulli, *Journal de Math.* (2) 1 (1856) 385–391.

22. KUMMER, E.E., Ueber eine allgemeine Eigenschaft der rationalen Entwickelung coëfficienten einer bestimmten Gattung analytischer Functionen, *J. Reine Angew. Math.* 41 (1851) 368–372.

23. LEHMER, D.H., Lacunary recurrence formulas for the numbers of Bernoulli and Euler, *Annals of Math.* 36 (1935) 637–649.

24. MEYER, G.E., Einige Beiträge zur Theorie der Bernoulli'schen Zahlen und der Secanten-Coefficienten, *Archiv der Mathematik und Physik* (1) 35 (1860) 449–474.

25. NIELSEN, NIELS, *Traité Élémentaire des Nombres de Bernoulli*, Gauthier-Villars, Paris (1923).

26. OHM, M., Etwas über die Bernoulli'schen Zahlen, *J. Reine Angew. Math.* 20 (1840) 11–12.

27. RADICKE, A., Die Recursions-formeln für die Berechnung die Bernoulli'schen und Euler'schen Zahlen, *Halle a. S. Louis Nebert.* (1880).

28. RAMANUJAN, SRINIVASA, Some properties of Bernoulli's numbers, *J. Indian Math. Soc.* 3 (1911) 219–234. Reprinted in *Collected Papers*, Chelsea, New York (1962).

29. RAMANUJAN, SRINIVASA, *Collected Papers*, Chelsea, New York (1962).

30. RAMANUJAN, SRINIVASA, *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay (1957).

31. RIESEL, HANS, Om rekursionsformler för Bernoullis tal, *Nordisk Matematisk Tidskrift* 9 (1961) 44–48.

32. SAALSCHÜTZ, L., *Vorlesungen über die Bernoullischen Zahlen*, Springer-Verlag, Berlin (1893).

33. SCHLÖMILCH, O., Ueber die rekurrirende Bestimmung der Bernoullischen Zahlen, *Archiv der Mathematik und Physik* (1) 3 (1843) 9–18.

34. SCHWERING, K., Zur Theorie der Bernoulli'schen Zahlen, *Math. Ann.* 52 (1899) 171–173.

35. VON STAUDT, K.G.C., Beweis eines Lehrsatzes die Bernoullischen Zahlen betreffend, *J. Reine Angew Math.* 21 (1840) 372–374.

36. STERN, M.A., Beiträge zur Theorie der Bernoulli'schen and Euler'schen Zahlen, *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen* 23 (1878), 44 pages.

37. STERN, M.A., Zur Theorie der Bernoulli'schen Zahlen, *J. Reine Angew. Math.* 88 (1880) 85–95.

38. USPENSKY, J.V. and M.A., HEASLET, *Elementary Number Theory*, McGraw-Hill, New York and London (1939).

39. WAGSTAFF, SAMUEL S., JR. The irregular primes to 125000, *Math. Comp.* 32 (1978) 583–591.

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