PRIMARY CARMICHAEL NUMBERS

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Abstract

Let $s_p(m)$ denote the sum of the base-p digits of the positive integer m. Kellner and Sondow defined a primary Carmichael number as a squarefree integer m with $s_p(m) = p$ for each prime divisor p of m. We show that the Prime k-tuples Conjecture implies that there are infinitely many primary Carmichael numbers. We define the degree of a Carmichael number and prove several results about this concept.

1. Introduction

A *Carmichael number* is a composite positive integer m for which the congruence $a^{m-1} \equiv 1 \pmod{m}$ holds for every integer a coprime to m. In 1889, Korselt proved that a composite integer m is a Carmichael number if and only if $p - 1 \mid m - 1$ for every prime divisor p of m. Carmichael proved that every Carmichael number is odd, squarefree, and has at least three prime factors.

Kellner and Sondow [4] found a new characterization of Carmichael numbers. Let $s_p(m)$ denote the sum of the base-p digits of the positive integer m. They proved that a positive integer m is a Carmichael number if and only if it is squarefree and each of its prime factors p satisfies both $s_p(m) \geq p$ and $s_p(m) \equiv 1 \pmod{p - 1}$. This characterization directly implies that m is odd and has at least three prime factors. They also defined a special type of Carmichael number they called a *primary Carmichael number*. This is a squarefree positive integer m with $s_p(m) = p$ for each of its prime factors p. Alford, Granville, and Pomerance [1] proved that there are infinitely many Carmichael numbers. Kellner and Sondow [4] counted the Carmichael numbers and primary Carmichael numbers up to 10^{10}, but were unable to prove that there are infinitely many primary Carmichael numbers. We prove below that the Prime k-tuples Conjecture implies that there are infinitely many primary Carmichael numbers.

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2. Chernick's polynomials

For integers $k \geq 3$ and $n \geq 1$ define

$$U_k(n) = (6n+1)(12n+1)\prod_{i=1}^{k-2}(9 \cdot 2^i n + 1).$$

Chernick [2] proved that $U_3(n)$ is a Carmichael number whenever all three of its factors $6n+1$, $12n+1$, and $18n+1$ are prime. He also showed that if $k \geq 4$ and 2^{k-4} divides n, then $U_k(n)$ is a Carmichael number whenever each of its k factors is prime.

Let $a_1, \ldots, a_k$ be positive integers and let $b_1, \ldots, b_k$ be nonzero integers. Let $P(x)$ denote the number of positive integers $n \leq x$ for which $a_i n + b_i$ is prime for each $i = 1, \ldots, k$. The Prime k-tuples Conjecture says that if no prime divides

$$\prod_{i=1}^{k}(a_i n + b_i) \tag{1}$$

for every n, then there is a constant $c > 0$ such that $P(x) \sim cx/\log^k x$ as $x \to \infty$. The Prime k-tuples Conjecture is supported by numerical data and a heuristic argument, but it has never been proved, except for $k = 1$.

Chernick [2] called a polynomial of the form of (1) *universal* if it is a Carmichael number for every n for which each of the k factors is prime. He gave many examples of universal polynomials, not just $U_k(n)$.

Since no prime can divide $U_k(n)$ for every n, the Prime k-tuples Conjecture and Chernick's theorem together imply that there are infinitely many Carmichael numbers with exactly k prime factors, in fact, at least $c_k x^{1/k}/\log^k x$ of them less than or equal to x for some $c_k > 0$.

3. Formulas for base-p digits

The smallest primary Carmichael number, mentioned by Kellner and Sondow [4], is Ramanujan's taxicab number 1729. This number also happens to be $U_3(1)$, which led us to the results in this section. Kellner [3] also noticed Chernick's paper and observed that $U_3(n)$ is a primary Carmichael number whenever all three of its factors are prime. In particular, he gave a different proof of Corollary 1 below.

Lemma 1. *Let n be a real number, $p = 6n+1$, $q = 12n+1$, $r = 18n+1$, and $m = U_3(n) = pqr$. Then*

$$m = 2p + (p-7)p^2 + 5p^3,$$
$$m = (3n)q + (9n+1)q^2, \text{ and}$$
$$m = (14n+1)r + (4n)r^2.$$

Proof. Write each equation in terms of n and check that it is an identity. The algebra becomes slightly simpler if one cancels the base prime from each side first. Alternately, one can use a computer algebra system to check the equations. $\square$

Corollary 1. *With the same hypotheses as Lemma 1, except that n is a positive integer, we have $s_p(m) = p$, $s_q(m) = q$, and $s_r(m) = r$.*

Proof. Since $p \geq 7$ (because $n \geq 1$), the coefficients 2, $p-7$, and 5 are between 0 and $p-1$, so they are the base-p digits of m. Thus $s_p(m) = 2+(p-7)+5 = p$. Similarly, the base-q and base-r digits lie in the correct intervals and $s_q(m) = 12n + 1 = q$ and $s_r(m) = 18n + 1 = r$. $\square$

Here is the promised theorem.

Theorem 1. *The Prime k-tuples Conjecture implies that there are infinitely many primary Carmichael numbers with exactly three prime factors.*

Proof. By the Prime k-tuples Conjecture, there are infinitely many positive integers n for which $U_3(n)$ has exactly three prime factors. By Corollary 1, each of these numbers $U_3(n)$ is a primary Carmichael number. $\square$

Can we get a similar result for $U_4(n)$ and find even more primary Carmichael numbers? Here are the corresponding lemma and corollary.

Lemma 2. *Let n be a real number, $p = 6n+1$, $q = 12n+1$, $r = 18n+1$, $t = 36n+1$, and $m = U_4(n) = pqrt$. Then*

$$m = (p - 10)p + 46p^2 + (p - 72)p^3 + 35p^4,$$
$$m = (6n + 1)q + (3n - 2)q^2 + (3n)q^3 + 2q^4,$$
$$m = (4n)r + (6n)r^2 + (8n + 1)r^3, \text{ and}$$
$$m = (26n + 1)t + (9n)t^2 + (n)t^3.$$

Proof. Write each equation in terms of n and check that it is an identity. $\square$

If $n = 1$, then $p = 7$ and $s_p(m) = 0 + 4 + 3 + 4 + 5 + 3 = 19 \neq 7$, so m is a Carmichael number but not primary. The next n for which $U_4(n)$ has only four prime factors is $n = 45$. But $m = U_4(45)$ is not a primary Carmichael number either, as this corollary shows.

Corollary 2. *With the same hypotheses as Lemma 2, except that $n \geq 12$ is an integer, we have $s_p(m) = 2p - 1$, $s_q(m) = q$, $s_r(m) = r$, and $s_t(m) = t$.*

Proof. Since $p \geq 73$ (because $n \geq 12$), the coefficients $p - 10$, 46, $p - 72$, and 35 are between 0 and $p - 1$, so they are the base-p digits of m. Thus $s_p(m) = (p - 10) + 46 + (p - 72) + 35 = 2p - 1$. Similarly, the base-$q$, base-$r$, and base-$t$

digits lie in the correct intervals, so $s_q(m) = 12n + 1 = q$, $s_r(m) = 18n + 1 = r$ and $s_t(m) = 36n + 1 = t$. □

Call a Carmichael number m *secondary* if m is not primary, but each prime factor p of m satisfies either $s_p(m) = p$ or $s_p(m) = 2p - 1$. Then we have this theorem.

Theorem 2. *The Prime k-tuples Conjecture implies that there are infinitely many secondary Carmichael numbers with exactly four prime factors.*

Proof. By the Prime k-tuples Conjecture, there are infinitely many positive integers n for which $U_4(n)$ has exactly four prime factors. By Corollary 2, each of these numbers $U_4(n)$ with $n \geq 12$ is a secondary Carmichael number. □

4. Numerical results

Using the online tables of Carmichael numbers computed by Pinch [5], we have counted the primary and secondary Carmichael numbers up to 10^{18}. Let $C(x)$, $C_1(x)$, and $C_2(x)$ denote the numbers of all, primary, and secondary Carmichael numbers up to x, respectively. Table 1 gives $C(x)$, $C_1(x)$, and $C_2(x)$ for $x = 10^n$, $n = 3, \ldots, 18$. It also shows the percentage of all Carmichael numbers that are primary or secondary. It appears that primary and secondary Carmichael numbers are rare among Carmichael numbers.

n	$C(10^n)$	$C_1(10^n)$	Percent	$C_2(10^n)$	Percent
3	1	0	0.00	0	0.00
4	7	2	28.57	2	28.57
5	16	4	25.00	6	37.50
6	43	9	20.93	17	39.53
7	105	19	18.10	42	40.00
8	255	51	20.00	74	29.02
9	646	107	16.56	152	23.53
10	1547	219	14.16	299	19.33
11	3605	417	11.57	547	15.17
12	8241	757	9.19	944	11.45
13	19279	1470	7.62	1671	8.67
14	44706	2666	5.96	3037	6.79
15	105212	5040	4.79	5346	5.08
16	246683	9280	3.76	9159	3.71
17	585355	17210	2.94	15570	2.66
18	1401644	32039	2.29	26216	1.87

Table 1: Number of Carmichael numbers below various limits.

The first secondary Carmichael number is 1105, which has three prime factors. The first secondary Carmichael number with four prime factors is 41041. The first Carmichael number which is neither primary nor secondary is 561, the smallest Carmichael number.

Recall that if a prime p divides a Carmichael number m, then $s_p(m) \geq p$ and $s_p(m) \equiv 1 \pmod{p-1}$. Define the *degree* of a Carmichael number m as the maximum of $(s_p(m)-1)/(p-1)$ taken over all prime factors p of m. Then the primary and secondary Carmichael numbers are those of degree 1 and 2, respectively. Table 2 shows the number of Carmichael numbers up to 10^{18} of each degree by number of prime factors.

Degree	3	4	5	6	7	8	9	10	11
1	31103	933	3	0	0	0	0	0	0
2	4339	15806	5918	153	0	0	0	0	0
3	144	13179	38497	16922	812	1	0	0	0
4	0	5931	46753	74231	24179	1188	4	0	0
5	0	2604	34517	97495	79757	15303	563	2	0
6	0	1113	22076	83500	107883	42789	4575	90	0
7	0	536	12954	58087	94741	54790	10084	478	3
8	0	279	7512	36386	65843	45254	11197	808	7
9	0	142	4085	20195	37923	27869	7854	679	18
10	0	77	2392	11637	20866	15808	4583	411	5
11	0	40	1441	7248	13023	9491	2807	254	5
12	0	17	898	4327	7711	5820	1683	158	5
13	0	12	488	2241	4211	3139	950	105	4
14	0	8	249	1116	2014	1496	423	50	1
15	0	2	106	508	856	592	190	15	1
16	0	1	61	223	321	242	48	6	0
17	0	1	37	131	139	74	11	1	0
18	0	2	31	80	87	48	7	1	0
19	0	1	21	80	86	30	4	0	0
20	0	1	16	45	54	21	5	0	0
21	0	0	6	25	31	23	2	0	0
22	0	0	2	24	9	9	2	0	0
23	0	0	0	3	7	7	0	0	0
24	0	0	0	2	0	1	0	0	0
25	0	0	0	1	0	1	1	0	0
26	0	0	0	0	0	1	0	0	0

Table 2: Carmichael numbers by degree and number of prime factors.

Roughly speaking, Carmichael numbers with more prime factors are more likely to have higher degrees. It may be true that all Carmichael numbers with only three factors have degree at most 3 and that Carmichael numbers with degree 1 have no more than five factors. These statements hold for the data up to 10^{18}.

5. What about $U_5(n)$ and beyond?

Can we use $U_5(n)$ to find infinitely many Carmichael numbers of degree 3? The complexity of the formulas for the digits of $U_k(n)$ increases as k increases.

Lemma 3. *Let n be a real number, $p = 6n+1$, $q = 12n+1$, $r = 18n+1$, $t = 36n+1$, $u = 72n+1$, and $m = U_5(n) = pqrtu$. Then*

$$m = (110)p + (p - 637)p^2 + (1355)p^3 + (p - 1260)p^4 + (431)p^5,$$
$$m = (6n - 2)q + (9n + 12)q^2 + (3n - 11)q^3 + (6n - 11)q^4 + (13)q^5,$$
$$m = (6n + 1)r + (16n)r^2 + (18n - 3)r^3 + (14n + 2)r^4 + (1)r^5,$$
$$m = (10n)t + (7n)t^2 + (17n + 1)t^3 + (2n)t^4, \text{ and}$$
$$m = (411(n/8) + 1)u + (143(n/8))u^2 + (21(n/8))u^3 + (n/8)u^4.$$

Corollary 3. *With the same hypotheses as Lemma 3, except that $n \geq 226$ is an integer and $8 \mid n$, we have $s_p(m) = 2p - 1$, $s_q(m) = 2q - 1$, $s_r(m) = 3r - 2$, $s_t(m) = t$, and $s_u(m) = u$.*

Only $s_u(m) = u$ requires $8 \mid n$; the others hold for all $n \geq 226$.

Proof. Since $p \geq 1355$ (because $n \geq 226$), the coefficients in the first equation in Lemma 3 are between 0 and $p - 1$, so they are the base-p digits of m. Thus $s_p(m) = 110+(p-637)+1355+(p-1260)+431 = 2p-1$. Similarly, the base-$q$, base-$r$, base-$t$, and base-$u$ digits lie in the correct intervals and $s_q(m) = 24n+1 = 2q-1$, $s_r(m) = 54n + 1 = 3r - 2$, $s_t(m) = 36n + 1 = t$, and $s_u(m) = (n/8)576n + 1 = 72n + 1 = u$. □

Theorem 3. *The Prime k-tuples Conjecture implies that there are infinitely many Carmichael numbers of degree 3 with exactly five prime factors.*

Proof. By the Prime k-tuples Conjecture, there are infinitely many positive integers n divisible by 8 for which $U_5(n)$ has exactly five prime factors. By Corollary 3, each of these numbers $U_5(n)$ with $n \geq 226$ is a Carmichael number of degree 3. □

Can we use $U_6(n)$ to find infinitely many Carmichael numbers of degree 4? Here are the results. The proofs are similar to those above.

Lemma 4. *Let n be a real number, $p = 6n+1$, $q = 12n+1$, $r = 18n+1$, $t = 36n+1$, $u = 72n+1$, $v = 144n+1$, and $m = U_6(n) = pqrtuv$. Then*

$$\begin{aligned}
m =&(p-2530)p + (17290)p^2 + (p-46476)p^3 + (61523)p^4 + \\
&\quad (p-40176)p^5 + (10367)p^6, \\
m =&(6n+28)q + (9n-159)q^2 + (3n+256)q^3 + (6n-2)q^4 + \\
&\quad (12n-283)q^5 + (161)q^6, \\
m =&(12n-4)r + (8n+9)r^2 + (2n+17)r^3 + (10n-40)r^4 + (4n+5)r^5 + (14)r^6, \\
m =&(6n+1)t + (19n-1)t^2 + (13n-2)t^3 + (26n+2)t^4 + (8n+1)t^5, \\
m =&(165(n/8))u + (103(n/8))u^2 + (265(n/8)+1)u^3 + (41(n/8))u^4 + (2(n/8))u^5, \\
&\quad \textit{and} \\
m =&(13119n/128+1)v + (4562n/128)v^2 + (704n/128)v^3 + \\
&\quad (46n/128)v^4 + (n/128)v^5.
\end{aligned}$$

Corollary 4. *With the same hypotheses as Lemma 4, except that $n \geq 7746$ is an integer and $128 \mid n$, we have $s_p(m) = 3p-2$, $s_q(m) = 3q-2$, $s_r(m) = 2r-1$, $s_t(m) = 2t-1$, $s_u(m) = u$, and $s_v(m) = v$.*

Thus the Carmichael numbers $U_6(n)$ have degree 3, not 4.

Theorem 4. *The Prime k-tuples Conjecture implies that there are infinitely many Carmichael numbers of degree 3 with exactly six prime factors.*

It turns out that $U_7(n)$ gives degree 4.

Lemma 5. *Let n be a real number, $a = n/8$, $b = n/128$, $c = n/4096$, $p = 6n+1$, $q = 12n+1$, $r = 18n+1$, $t = 36n+1$, $u = 72n+1$, $v = 144n+1$, $w = 288n+1$, and $m = U_7(n) = pqrtuvw$. Then*

$$\begin{aligned}
m =&118910p + (p-934117)p^2 + 3014339p^3 + (p-5122476)p^4 + 4841423p^5 + \\
&\quad (p-2415744)p^6 + 497663p^7, \\
m =&(6n-632)q + (9n+4323)q^2 + (3n-9722)q^3 + (6n+6208)q^4 + 6466q^5 + \\
&\quad (12n-10529)q^6 + 3887q^7, \\
m =&70r + (18n-212)r^2 + (8n-113)r^3 + (8n+884)r^4 + (10n-727)r^5 + \\
&\quad (10n-128)r^6 + 227r^7, \\
m =&(30n-5)t + (23n+16)t^2 + (25n+2)t^3 + (30n-26)t^4 + (8n+2)t^5 + \\
&\quad (28n+11)t^6 + t^7, \\
m =&(81a+1)u + (351a-1)u^2 + (193a-2)u^3 + (361a+2)u^4 + \\
&\quad (158a+1)u^5 + (8a)u^6, \\
m =&(5313b)v + (3244b)v^2 + (8420b+1)v^3 + (1362b)v^4 + (91b)v^5 + (2b)v^6, \ \textit{and} \\
m =&(839133c+1)w + (291717c)w^2 + (45506c)w^3 + (3194c)w^4 + (97c)w^5 + (c)w^6.
\end{aligned}$$

Corollary 5. *With the same hypotheses as Lemma 5, except that* $n \geq 853746$ *is an integer and* $4096 \mid n$*, we have* $s_p(m) = 3p - 2$*,* $s_q(m) = 3q - 2$*,* $s_r(m) = 3r - 2$*,* $s_t(m) = 4t - 3$*,* $s_u(m) = 2u - 1$*,* $s_v(m) = v$*, and* $s_w(m) = w$*.*

Theorem 5. *The Prime k-tuples Conjecture implies that there are infinitely many Carmichael numbers of degree 4 with exactly seven prime factors.*

Perhaps one can continue these arguments to show that the Prime k-tuples Conjecture implies that there are infinitely many Carmichael numbers of each degree greater than or equal to 1.

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References

[1] W. R. Alford, A. Granville, and C. Pomerance, There are infinitely many Carmichael numbers, *Ann. Math.* **139** (1994), 703-722.

[2] J. Chernick, On Fermat's simple theorem, *Bull. Amer. Math. Soc.* **45**(1939), 269-274.

[3] B. C. Kellner, On primary Carmichael numbers, *Integers* **22** (2022), #A38, 1-39.

[4] B. C. Kellner and J. Sondow, On Carmichael and polygonal numbers, Bernoulli polynomials, and sums of base-p digits, *Integers* **21** (2021), #A52, 1-21.

[5] R. G. E. Pinch, Table of Carmichael numbers to 10^{18}, February, 2012. Available at the url `http://www.s369624816.websitehome.co.uk/rgep/index.html` .